Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **36**, 1–15; doi: 10.14232/ejqtde.2015.1.36 http://www.math.u-szeged.hu/ejqtde/

A Perron-type theorem for nonautonomous difference equations with nonuniform behavior

Hailong Zhu ✉ [1], **Can Zhang**[2] and **Yongxin Jiang**[2]

[1]School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu 233030, China
[2]Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China

Received 2 October 2014, appeared 7 July 2015
Communicated by Zuzana Došlá

Abstract. We show that if the Lyapunov exponents of a linear difference equation are limits, then the same happens with the Lyapunov exponents of the solutions of the nonlinear equations for any sufficiently small nonlinear perturbations. We consider the case with a very general nonuniform behavior, which is called nonuniform (h, k, μ, ν) behavior.

Keywords: (h, k, μ, ν)-dichotomies, h-Lyapunov exponent, nonautonomous difference equations.

2010 Mathematics Subject Classification: 34D08, 34D09.

1 Introduction

We say that an increasing function $h\colon \mathbb{N} \to (0, +\infty)$ is a *growth rate* if

$$\lim_{m\to+\infty} h(m) = +\infty.$$

For example, e^{am}, $m^a + b$, $m^a e^{bm}$, $m^a \log(b + m)$ with $a, b > 0$ are growth rates. Given a growth rate h, we can define *h-Lyapunov exponent* $\lambda\colon \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ associated with the linear difference equation

$$x_{m+1} = A_m x_m, \qquad m \in \mathbb{N} \tag{1.1}$$

by

$$\lambda(x) = \limsup_{m\to+\infty} \frac{\log\|\mathcal{A}(m, 1)x\|}{\log h(m)}, \tag{1.2}$$

where $x \in \mathbb{C}^n$, $(A_m)_{m\in\mathbb{N}}$ is a sequence of $n \times n$ invertible matrices with complex entries such that

$$\sup_{m\in\mathbb{N}} \|A_m\| < +\infty,$$

✉ Corresponding author. Email: hai-long-zhu@163.com

and $\mathcal{A}$ is the cocycle produced by $(A_m)_{m\in\mathbb{N}}$, that is

$$\mathcal{A}(m,l)=\begin{cases} A_{m-1}\cdots A_l, & \text{if } m>l,\\ \mathrm{Id}, & \text{if } m=l,\\ A_m^{-1}\cdots A_{l-1}^{-1}, & \text{if } m<l. \end{cases}$$

In this paper, we show that if all h-Lyapunov exponents of (1.1) are limits, the asymptotic behavior of (1.1) persists under sufficiently small perturbations for the nonlinear equation

$$x_{m+1}=A_m x_m+f_m(x_m), \tag{1.3}$$

where the perturbation $f_m\colon \mathbb{C}^m\to\mathbb{C}^m$ is continuous and small enough. More precisely, if the sequence (1.3) is not eventually zero, the limit

$$\lambda=\lim_{m\to+\infty}\frac{\log\|x_m\|}{\log h(m)}$$

exists and coincides with an h-Lyapunov exponent of (1.1). The required smallness of the perturbation is that

$$\sum_{m=1}^{\infty}\mu(m+1)^{\delta_1}\nu(m+1)^{\delta_2}\sup_{x\neq 0}\frac{\|f_m(x)\|}{\|x\|}<+\infty, \tag{1.4}$$

or simply

$$\sum_{m=1}^{\infty}\mu(m+1)^{\delta_1}\nu(m+1)^{\delta_2}\frac{\|f_m(x_m)\|}{\|x_m\|}<+\infty$$

for some $\delta_1,\delta_2>0$, where μ,ν are two given growth rates. When $\mu(m)=\nu(m)=e^m$, we recover the result in [9] and (1.4) becomes

$$\sum_{m=1}^{\infty}e^{\delta m}\sup_{x\neq 0}\frac{\|f_m(x)\|}{\|x\|}<+\infty,$$

for some $\delta>0$.

In the literature, the results related to the above problems are called "Perron-type theorems". For the case $A_m=A$ being constant, the results were proved by Coffman [13]. A related result for perturbations of a differential equation $x'=Ax$ with constant coefficient can be found in the book [14]. More results can be found in [15–19,22,23]. Recently, Barreira and Valls established the Perron-type theorems for nonautonomous differential equations [8] and nonautonomous difference equations [7,9,10], based on Lyapunov's theory of regularity. Especially, they considered the cases with nonuniform exponential behavior. In this paper, we will follow the ideas of Barreira and Valls.

Such problems are also very close to the theory of nonuniform exponential dichotomies, which was inspired both by the classical notion of exponential dichotomy and by the notion of nonuniformly hyperbolic trajectory introduced by Barreira and Pesin (see [3]), and have been developed in a systematic way by Barreira and Valls (see [4–6] and the references therein) during the last several years. As explained by Barreira and Valls, in comparison to the notion of exponential dichotomies introduced by Perron in [21], nonuniform exponential dichotomy is a useful and weaker notion. A very general type of nonuniform exponential dichotomy, the so-called (μ,ν) exponential dichotomy, has been considered in [1,2,11,12].

Compared with those results in the literature, the novelty of this work is that we establish the Perron-type theorem for nonautonomous difference equations with different growth rates in the uniform parts and nonuniform parts. More precisely, we consider the (h,k,μ,ν) nonuniform behavior and this creates additional complications in the analysis. We refer the reader to [20] for some results on the so-called (h,k)-dichotomies, which were introduced by Pinto.

2 Preliminaries

Given a growth rate h and consider a sequence $(A_m)_{m\in\mathbb{N}}$ of invertible $n\times n$ matrices with complex entries such that

$$\limsup_{m\to+\infty}\frac{\log\|\mathcal{A}(m,1)\|}{\log h(m)}<+\infty. \tag{2.1}$$

The h-Lyapunov exponent $\lambda\colon\mathbb{C}^n\to\mathbb{R}\cup\{-\infty\}$ of equation (1.1) is defined by the formula (1.2), with the convention that $\log 0=-\infty$ to illustrate the value $\lambda(0)=-\infty$. It follows from (2.1) that λ never takes the value $+\infty$. By the general theory of Lyapunov exponents (see [3] for details), we know that the Lyapunov exponent λ can take on only finitely many distinct values $-\infty\le\lambda_1<\cdots<\lambda_p$, where $p\le n$. Furthermore, for each $1\le i\le p$, we define

$$E_i=\{x\in\mathbb{C}^n:\lambda(x)\le\lambda_i\}$$

as a linear subspace over $\mathbb{C}^n$ (with the convention that $E_0=\{0\}$). Obviously,

$$\{0\}=E_0\subsetneq E_1\subsetneq\cdots\subsetneq E_p.$$

We set $k_i=\dim E_i-\dim E_{i-1}$.

Now we describe the assumptions in the paper.

(H1) There exist decompositions

$$\mathbb{C}^n=F_m^1\oplus F_m^2\oplus\cdots\oplus F_m^p,\qquad m\in\mathbb{N}$$

into subspaces of dimension $\dim F_m^i=k_i$ such that for each $m,l\in\mathbb{N}$ and $i=1,\dots,p$,

$$\mathcal{A}(m,l)F_l^i=F_m^i.$$

Thus for a given number $b\in\mathbb{R}$ which is not a h-Lyapunov exponent, there exist a decomposition

$$\mathbb{C}^n=E_l\oplus F_l, \tag{2.2}$$

and $E_l=E_i$ when $\lambda_i<b<\lambda_{i+1}$, where

$$E_l=\bigoplus_{\lambda_i<b}F_l^i\quad\text{and}\quad F_l=\bigoplus_{\lambda_i>b}F_l^i$$

for each $l\in\mathbb{N}$.

(H2) Take $a<b<c$ such that the interval $[a,c]$ contains no Lyapunov exponent and a given constant $\varepsilon>0$, there exists a constant $K=K(\varepsilon)>0$ such that

$$\|\mathcal{A}(m,l)P_l\|\le K\left(\frac{h(m)}{h(l)}\right)^a\mu(l)^\varepsilon,\qquad m\ge l, \tag{2.3}$$

and
$$\|\mathcal{A}(m,l)Q_l\| \le K\left(\frac{k(m)}{k(l)}\right)^c \nu(l)^\varepsilon, \qquad m \le l, \tag{2.4}$$
in which P_l and Q_l are the projections associated with the decomposition (2.2) and h, k, μ, ν are growth rates.

(H3) The growth rates h, k, μ, ν satisfy
$$\mu(m) \le h(m), \qquad \mu(m) \le k(m), \qquad \nu(m) \le h(m), \qquad \nu(m) \le k(m), \qquad m \in \mathbb{N},$$
and h, k satisfy the compensation condition: there exists a constant $0 < \eta < 1$ such that
$$\left(\frac{h(m)}{h(l)}\right)^a \le \eta \left(\frac{k(m)}{k(l)}\right)^c, \qquad m \ge l \ge 0.$$

Take $m = l$ in (2.3)-(2.4), we can obtain
$$\|P_m\| \le K\mu(m)^\varepsilon \quad \text{and} \quad \|Q_m\| \le K\nu(m)^\varepsilon. \tag{2.5}$$
Moreover, for every $m, l \in \mathbb{N}$, we have
$$P_m\mathcal{A}(m,l) = \mathcal{A}(m,l)P_l, \qquad Q_m\mathcal{A}(m,l) = \mathcal{A}(m,l)Q_l. \tag{2.6}$$

The compensation condition in H3 is very important in our analysis. For the uniform (h,k) behavior, [20, Section VI] illustrate importance of "h and k are compensated".

In Section 4, we will give two explicit examples of sequences $(A_m)_{m\in\mathbb{N}}$ which satisfy assumptions (H1)–(H3).

3 Main results

The following is our main result. It claims that under sufficiently small perturbations, the Lyapunov exponent of (1.3) coincides with some Lyapunov exponent of the unperturbed difference equation (1.1).

Theorem 3.1. *Let $(x_m)_{m\in\mathbb{N}}$ be a sequence satisfying* (1.3) *and*
$$\|f_m(x_m)\| \le \gamma_m\|x_m\|, \qquad m \in \mathbb{N}, \tag{3.1}$$
where the sequence γ_m satisfies
$$\sum_{m=1}^{\infty} \mu(m+1)^{\delta_1}\nu(m+1)^{\delta_2}\gamma_m < +\infty \tag{3.2}$$
for some $\delta_1, \delta_2 \ge \varepsilon > 0$ and two growth rates μ, ν are given in (H2). *Assume that conditions* (H1)–(H3) *are satisfied. Then one of the following alternatives hold:*

(1) *$x_m = 0$ for all sufficiently large m;*

(2) *the limit*
$$\lim_{m\to\infty} \frac{\log\|x_m\|}{\log h(m)}$$
exists and coincides with a Lyapunov exponent of (1.1).

Before presenting the proof of Theorem 3.1, we prove several lemmas.

Lemma 3.2. *There exists a constant $K' > 0$ such that*

$$\|x_m\| \le K' \left(\frac{h(m)}{h(l)}\right)^d \mu(l)^\varepsilon \|x_l\| \tag{3.3}$$

for every $m, l \in \mathbb{N}$ with $m \ge l$ and $d > \lambda_p$. In particular, given $r \in \mathbb{N}$ there exists $C = C(r) > 0$ such that

$$C^{-1}\mu((s+1)r)^{-\varepsilon}\|x_{(s+1)r}\| \le \|x_m\| \le C\mu(sr)^\varepsilon \|x_{sr}\| \tag{3.4}$$

for all $l \le sr \le m \le (s+1)r$.

Proof. For each $m \ge l$, (1.3) has a solution x_m which can be written as

$$x_m = \mathcal{A}(m,l)x_l + \sum_{j=l}^{m-1} \mathcal{A}(m,j+1)f_j(x_j). \tag{3.5}$$

Note that $d > \lambda_p$, it follows from (2.3) that

$$\|\mathcal{A}(m,l)\| \le K\left(\frac{h(m)}{h(l)}\right)^d \mu(l)^\varepsilon, \qquad m \ge l. \tag{3.6}$$

Then by (3.1) and (3.5), we obtain

$$\|x_m\| \le K\left(\frac{h(m)}{h(l)}\right)^d \mu(l)^\varepsilon \|x_l\| + K\sum_{j=l}^{m-1}\left(\frac{h(m)}{h(j+1)}\right)^d \mu(j+1)^\varepsilon \gamma_j \|x_j\|,$$

and hence,

$$\left(\frac{h(m)}{h(l)}\right)^{-d}\|x_m\| \le K\mu(l)^\varepsilon\|x_l\| + K\sum_{j=l}^{m-1}\left(\frac{h(l)}{h(j+1)}\right)^d \mu(j+1)^\varepsilon\gamma_j\|x_j\|.$$

One can use induction to show that

$$\left(\frac{h(m)}{h(l)}\right)^{-d}\|x_m\| \le K\mu(l)^\varepsilon\|x_l\| \prod_{j=l}^{m-1}(1 + K\mu(j+1)^\varepsilon\gamma_j)$$

for $m \ge l$. Hence

$$\begin{aligned}\|x_m\| &\le K\left(\frac{h(m)}{h(l)}\right)^d \mu(l)^\varepsilon\|x_l\| \exp\left(\sum_{j=l}^{m-1} K\mu(j+1)^\varepsilon\gamma_j\right)\\ &\le K\left(\frac{h(m)}{h(l)}\right)^d \mu(l)^\varepsilon\|x_l\| \exp\left(K\sum_{j=1}^{\infty}\mu(j+1)^\varepsilon\gamma_j\right).\end{aligned}$$

Therefore, by using (3.2), we know that property (3.3) holds with

$$K' = K\exp\left(K\sum_{j=1}^{\infty}\mu(j+1)^\varepsilon\gamma_j\right).$$

In particular, (3.3) implies that property (3.4) holds with $C = K'(\frac{A}{B})^d$, where

$$A = \max_{sr \le t \le (s+1)r} h(t), \qquad B = \min_{sr \le t \le (s+1)r} h(t).$$

This completes the proof of the lemma. ∎

Let $b \in \mathbb{R}$ be a number that is not an h-Lyapunov exponent. Let also $a < b < c$ be as in Section 2. We consider the norm

$$\|x\|_m = \sup_{m' \geq m} \left((\tfrac{h(m')}{h(m)})^{-a} \|\mathcal{A}(m', m) P_m x\| \right) + \sup_{m' \leq m} \left((\tfrac{k(m')}{k(m)})^{-c} \|\mathcal{A}(m', m) Q_m x\| \right)$$

for each $m \in \mathbb{N}$ and $x \in \mathbb{C}^n$. We have

$$\|x\|_m = \|P_m x\|_m + \|Q_m x\|_m \tag{3.7}$$

and one can easily verify that

$$\|x\| \leq \|x\|_m \leq K(\mu(m)^{\varepsilon} + \nu(m)^{\varepsilon})\|x\|. \tag{3.8}$$

Lemma 3.3. *We have*

$$\|\mathcal{A}(m, l) P_l x\|_m \leq \left(\frac{h(m)}{h(l)} \right)^a \|P_l x\|_l \quad \text{for } m \geq l, \tag{3.9}$$

and

$$\|\mathcal{A}(m, l) Q_l x\|_m \geq \left(\frac{k(m)}{k(l)} \right)^c \|Q_l x\|_l \quad \text{for } m \geq l. \tag{3.10}$$

Proof. For $m \geq l$ we have

$$\begin{aligned}
\|\mathcal{A}(m, l) P_l x\|_m &= \sup_{m' \geq m} \left(\left(\frac{h(m')}{h(m)} \right)^{-a} \|\mathcal{A}(m', m) \mathcal{A}(m, l) P_l x\| \right) \\
&= \left(\frac{h(m)}{h(l)} \right)^a \sup_{m' \geq m} \left(\left(\frac{h(m')}{h(l)} \right)^{-a} \|\mathcal{A}(m', l) P_l x\| \right) \\
&\leq \left(\frac{h(m)}{h(l)} \right)^a \sup_{m' \geq l} \left(\left(\frac{h(m')}{h(l)} \right)^{-a} \|\mathcal{A}(m', l) P_l x\| \right) \\
&= \left(\frac{h(m)}{h(l)} \right)^a \|P_l x\|_l.
\end{aligned}$$

Similarly, for $m \geq l$ we have

$$\begin{aligned}
\|\mathcal{A}(m, l) Q_l x\|_m &= \sup_{m' \leq m} \left(\left(\frac{k(m')}{k(m)} \right)^{-c} \|\mathcal{A}(m', m) \mathcal{A}(m, l) Q_l x\| \right) \\
&= \left(\frac{k(m)}{k(l)} \right)^c \sup_{m' \leq m} \left(\left(\frac{k(m')}{k(l)} \right)^{-c} \|\mathcal{A}(m', l) Q_l x\| \right) \\
&\geq \left(\frac{k(m)}{k(l)} \right)^c \sup_{m' \leq l} \left(\left(\frac{k(m')}{k(l)} \right)^{-c} \|\mathcal{A}(m', l) Q_l x\| \right) \\
&= \left(\frac{k(m)}{k(l)} \right)^c \|Q_l x\|_l.
\end{aligned}$$

This completes the proof of the lemma. □

Now let $(x_m)_{m \in \mathbb{N}}$ be a sequence satisfying (1.3). Using the decomposition in (2.2), we can write $x_m = y_m + z_m$, where

$$y_m = P_m x_m, \qquad z_m = Q_m x_m.$$

Applying P_m and Q_m to both sides of (3.5) and using (2.6), we obtain,

$$y_m = \mathcal{A}(m,l)y_l + \sum_{j=l}^{m-1} \mathcal{A}(m,j+1)P_{j+1}f_j(x_j),$$

and

$$z_m = \mathcal{A}(m,l)z_l + \sum_{j=l}^{m-1} \mathcal{A}(m,j+1)Q_{j+1}f_j(x_j).$$

Lemma 3.4. *Let $b \in \mathbb{R}$ be a number that is not an h-Lyapunov exponent, then one of the following alternatives holds:*

1.
$$\limsup_{m\to+\infty} \frac{\log \|x_m\|}{\log h(m)} < b \tag{3.11}$$

and

$$\lim_{s\to+\infty} \frac{\|z_{sr}\|_{sr}}{\|y_{sr}\|_{sr}} = 0; \tag{3.12}$$

2.
$$\liminf_{m\to+\infty} \frac{\log \|x_m\|}{\log k(m)} > b \tag{3.13}$$

and

$$\lim_{s\to+\infty} \frac{\|y_{sr}\|_{sr}}{\|z_{sr}\|_{sr}} = 0. \tag{3.14}$$

Proof. For $m \geq sr$ we have

$$y_m = \mathcal{A}(m,sr)P_{sr}x_{sr} + \sum_{j=sr}^{m-1} \mathcal{A}(m,j+1)P_{j+1}f_j(x_j) \tag{3.15}$$

and

$$z_m = \mathcal{A}(m,sr)Q_{sr}x_{sr} + \sum_{j=sr}^{m-1} \mathcal{A}(m,j+1)Q_{j+1}f_j(x_j). \tag{3.16}$$

By (3.8) and (3.10), it follows from (3.16) that for $m \geq sr$

$$\begin{aligned}\|z_m\|_m &\geq \|\mathcal{A}(m,sr)Q_{kr}x_{sr}\|_m - \left\| \sum_{j=sr}^{m-1} \mathcal{A}(m,j+1)Q_{j+1}f_j(x_j) \right\|_m \\ &\geq \left(\frac{k(m)}{k(sr)}\right)^c \|z_{sr}\|_{sr} - K(\mu(m)^\varepsilon + \nu(m)^\varepsilon) \sum_{j=sr}^{m-1} \|\mathcal{A}(m,j+1)Q_{j+1}f_j(x_j)\|.\end{aligned}$$

Using (2.5), (3.4), (3.6), (3.7) and (3.8), it follows from (H3) that for $m \leq (s+1)r$,

$$\begin{aligned}\|z_m\|_m &\geq \left(\frac{k(m)}{k(sr)}\right)^c \|z_{sr}\|_{sr} - K^3(\mu(m)^\varepsilon + \nu(m)^\varepsilon) \sum_{j=sr}^{m-1} \left(\frac{h(m)}{h(j+1)}\right)^d \mu(j+1)^\varepsilon \nu(j+1)^\varepsilon \gamma_j \|x_j\| \\ &\geq \left(\frac{k(m)}{k(sr)}\right)^c \|z_{sr}\|_{sr} - D_1\tau_s^1 \|x_{sr}\| \\ &\geq \left(\frac{k(m)}{k(sr)}\right)^c \|z_{sr}\|_{sr} - D_1\tau_s^1 (\|y_{sr}\|_{sr} + \|z_{sr}\|_{sr})\end{aligned} \tag{3.17}$$

with

$$D_1 = K^3 C\mu(sr)^{\varepsilon} \max_{sr\le m\le (s+1)r} (\mu(m)^{\varepsilon}+\nu(m)^{\varepsilon}) \max_{sr\le j\le m-1} \left(\frac{h(m)}{h(j+1)}\right)^{d}$$

and

$$\tau_s^1 = \sum_{j=sr}^{(s+1)r-1} \mu(j+1)^{\varepsilon}\nu(j+1)^{\varepsilon}\gamma_j.$$

By (3.2),

$$\tau_s^1 \to 0, \qquad s\to\infty. \tag{3.18}$$

Using (2.3), (3.4), (3.7), (3.8), (3.9) and (3.15), it follows from similar estimates that for $sr\le m\le (s+1)r$,

$$\begin{aligned}\|y_m\|_m &\le \left(\frac{h(m)}{h(sr)}\right)^{a}\|y_{sr}\|_{sr} + K^2(\mu(m)^{\varepsilon}+\nu(m)^{\varepsilon})\sum_{j=sr}^{(s+1)r-1}\left(\frac{h(m)}{h(j+1)}\right)^{a}\mu(j+1)^{\varepsilon}\gamma_j\|x_j\|\\ &\le \left(\frac{h(m)}{h(sr)}\right)^{a}\|y_{sr}\|_{sr} + D_2\tau_s^2\|x_{sr}\|\\ &\le \left(\frac{h(m)}{h(sr)}\right)^{a}\|y_{sr}\|_{sr} + D_2\tau_s^2(\|y_{sr}\|_{sr}+\|z_{sr}\|_{sr}),\end{aligned} \tag{3.19}$$

with

$$D_2 = K^2 C\mu(sr)^{\varepsilon} \max_{sr\le m\le (s+1)r} (\mu(m)^{\varepsilon}+\nu(m)^{\varepsilon}) \max_{sr\le j\le m-1} \left(\frac{h(m)}{h(j+1)}\right)^{a}$$

and

$$\tau_s^2 = \sum_{j=sr}^{(s+1)r-1} \mu(j+1)^{\varepsilon}\gamma_j.$$

By (3.2),

$$\tau_s^2 \to 0, \qquad s\to\infty. \tag{3.20}$$

Inequalities (3.17) and (3.19) yield that

$$\|z_m\|_m \ge \alpha\|z_{sr}\|_{sr} - D\tau_s^1(\|y_{sr}\|_{sr}+\|z_{sr}\|_{sr}) \tag{3.21}$$

and

$$\|y_m\|_m \le \beta\|y_{sr}\|_{sr} + D\tau_s^2(\|y_{sr}\|_{sr}+\|z_{sr}\|_{sr}) \tag{3.22}$$

with

$$\alpha = \left(\frac{k(m)}{k(sr)}\right)^{c}, \qquad \beta = \left(\frac{h(m)}{h(sr)}\right)^{a} \quad \text{and} \quad D = D_1 + D_2.$$

Now we claim that either

$$\|z_{sr}\|_{sr} \le \|y_{sr}\|_{sr} \quad \text{for all large } s, \tag{3.23}$$

or

$$\|y_{sr}\|_{sr} < \|z_{sr}\|_{sr} \quad \text{for all large } s. \tag{3.24}$$

We shall show that if (3.23) fails, then (3.24) holds. Let us assume that (3.23) does not hold. Then

$$\|y_{sr}\|_{sr} < \|z_{sr}\|_{sr} \quad \text{for infinitely many } s. \tag{3.25}$$

By (3.18) and (3.20), given $\tau > 0$, there exists s' such that $\tau_s^1, \tau_s^2 < \tau$ for $s \geq s'$. By (3.21) and (3.22), we find that for infinitely many integers $s \geq s'$,

$$\|z_{(s+1)r}\|_{(s+1)r} \geq (\alpha_s - D\tau)\|z_{sr}\|_{sr} - D\tau\|y_{sr}\|_{sr} \tag{3.26}$$

and

$$\|y_{(s+1)r}\|_{(s+1)r} \leq (\beta_s + D\tau)\|y_{sr}\|_{sr} + D\tau\|z_{sr}\|_{sr} \tag{3.27}$$

with

$$\alpha_s = \left(\frac{k((s+1)r)}{k(sr)}\right)^c, \qquad \beta_s = \left(\frac{h((s+1)r)}{h(sr)}\right)^a \quad \text{and} \quad D = D_1 + D_2.$$

By (3.25), there exists $s'' > s'$ such that

$$\|y_{s''r}\|_{s''r} < \|z_{s''r}\|_{s''r}.$$

We show by induction on s that

$$\|y_{sr}\|_{sr} < \|z_{sr}\|_{sr} \quad \text{for all } s \geq s''. \tag{3.28}$$

Let us assume that $\|y_{sr}\|_{sr} < \|z_{sr}\|_{sr}$ for some $s \geq s''$. By (3.26) and (3.27), we have

$$\|z_{(s+1)r}\|_{(s+1)r} \geq (\alpha_s - 2D\tau)\|z_{sr}\|_{sr}$$

and

$$\|y_{(s+1)r}\|_{(s+1)r} \leq (\beta_s + 2D\tau)\|z_{sr}\|_{sr}$$

provided that τ is sufficiently small.

Under the assumption (H3), it is easy to see that

$$\|y_{(s+1)r}\|_{(s+1)r} \leq \frac{\beta_s + 2D\tau}{\alpha_s - 2D\tau}\|z_{(s+1)r}\|_{(s+1)r} < \|z_{(s+1)r}\|_{(s+1)r}.$$

This shows that (3.28) holds. Thus, we show that if (3.23) fails, then (3.24) holds. As a consequence, we have the following two cases.

Case 1. Let us assume that (3.23) holds. We show that (3.11) and (3.12) hold.

Given $\tau > 0$, there exists s_0 such that $\tau_s^1, \tau_s^2 < \tau$ and $\|z_{sr}\|_{sr} \leq \|y_{sr}\|_{sr}$ for $s \geq s_0$. By (3.27), we find that for $s \geq s_0$,

$$\|y_{(s+1)r}\|_{(s+1)r} \leq (\beta_s + 2D\tau)\|y_{sr}\|_{sr},$$

which implies that

$$\|y_{sr}\|_{sr} \leq \|y_{s_0r}\|_{s_0r} \prod_{j=s_0}^{s-1}(\beta_j + 2D\tau).$$

Together with (3.4),(3.7) and (3.8), this yields that for for $s \geq s_0$ and $sr \leq m \leq (s+1)r$,

$$\begin{aligned}
\|x_m\| &\leq C\mu(sr)^{\varepsilon}\|x_{sr}\| \leq C\mu(sr)^{\varepsilon}\|x_{sr}\|_{sr} \\
&= C\mu(sr)^{\varepsilon}(\|y_{sr}\|_{sr} + \|z_{sr}\|_{sr}) \\
&\leq 2C\mu(sr)^{\varepsilon}\|y_{sr}\|_{sr} \\
&\leq 2C\|y_{s_0r}\|_{s_0r}\mu(sr)^{\varepsilon} \prod_{j=s_0}^{s-1}(\beta_j + 2D\tau).
\end{aligned}$$

Under the assumptions (H3), thus we have

$$\limsup_{m\to+\infty}\frac{\log\|x_m\|}{\log h(m)}\le\limsup_{s\to+\infty}\left(\frac{\log\prod_{j=s_0}^{s-1}(\beta_j+2D\tau)}{\log h(sr)}+\varepsilon\right).$$

Since τ is arbitrary and provided that ε is sufficiently small, we obtain

$$\begin{aligned}\limsup_{m\to+\infty}\frac{\log\|x_m\|}{\log h(m)}&\le\limsup_{s\to+\infty}\left(\frac{\log\prod_{j=s_0}^{s-1}\beta_j}{\log h(sr)}+\varepsilon\right)\\&=a+\varepsilon<b.\end{aligned}$$

This establishes (3.11). Now we prove (3.12). We know that $\|y_{sr}\|_{sr}>0$ for all large s, since otherwise (3.7) and (3.23) yield

$$\|x_m\|\le C\mu(sr)^{\varepsilon}\|x_{sr}\|_{sr}\le 2C\mu(sr)^{\varepsilon}\|y_{sr}\|_{sr}=0$$

for all large m, contradicting the hypothesis that $x_m>0$ exists for some large m.

Define

$$S=\limsup_{s\to+\infty}\frac{\|z_{sr}\|_{sr}}{\|y_{sr}\|_{sr}}.$$

By (3.23) we have $0\le S\le 1$. It follows from (3.22) and (3.23) that for all large s

$$\|y_{(s+1)r}\|_{(s+1)r}\le(\beta_s+2D\tau_s^2)\|y_{sr}\|_{sr}.$$

Together with (3.21), this yields that for all large s,

$$\frac{\|z_{(s+1)r}\|_{(s+1)r}}{\|y_{(s+1)r}\|_{(s+1)r}}\ge\frac{\alpha_s-D\tau_s^1}{\beta_s+2D\tau_s^2}\frac{\|z_{sr}\|_{sr}}{\|y_{sr}\|_{sr}}-\frac{D\tau_s^1}{\beta_s+2D\tau_s^2}.$$

Taking the limit superior on both sides and using (3.18) and (3.20), we obtain $S\ge(\alpha_s/\beta_s)S$. Under the assumption (H3) we have

$$\lim_{s\to\infty}\frac{\alpha_s}{\beta_s}\ge\frac{1}{\eta}>1,$$

this implies $S=0$, and (3.12) holds.

Case 2. Now we assume that (3.24) holds. We show that (3.13) and (3.14) hold.

Given $\tau>0$, there exists s_0 such that $\tau_s^1,\tau_s^2<\tau$ and $\|y_{sr}\|_{sr}<\|z_{sr}\|_{sr}$ for $s\ge s_0$. By (3.26), we find that for $s\ge s_0$,

$$\|z_{(s+1)r}\|_{(s+1)r}\ge(\alpha_s-2D\tau)\|z_{sr}\|_{sr},$$

which implies that

$$\|z_{(s+1)r}\|_{(s+1)r}\ge\|z_{s_0r}\|_{s_0r}\prod_{j=s_0}^{s}(\alpha_j-2D\tau).$$

Together with (3.4), (3.7) and (3.8), this yields that for for $s\ge s_0$ and $sr\le m\le(s+1)r$,

$$\begin{aligned}\|x_m\|&\ge C^{-1}\mu((s+1)r)^{-\varepsilon}\|x_{(s+1)r}\|\\&\ge C^{-1}K^{-1}(\mu((s+1)r)^{\varepsilon}+\nu((s+1)r)^{\varepsilon})^{-1}\mu((s+1)r)^{-\varepsilon}\|x_{(s+1)r}\|_{(s+1)r}\\&\ge C^{-1}K^{-1}(\mu((s+1)r)^{\varepsilon}+\nu((s+1)r)^{\varepsilon})^{-1}\mu((s+1)r)^{-\varepsilon}\|z_{(s+1)r}\|_{(s+1)r}\\&\ge\frac{\|z_{s_0r}\|_{s_0r}}{CK(\mu((s+1)r)^{\varepsilon}+\nu((s+1)r)^{\varepsilon})\mu((s+1)r)^{\varepsilon}}\prod_{j=s_0}^{s}(\alpha_j-2D\tau).\end{aligned}$$

Under the assumptions (H2), thus we have

$$\liminf_{m\to+\infty}\frac{\log\|x_m\|}{\log k(m)}\geq\liminf_{s\to+\infty}\left(\frac{\log\prod_{j=s_0}^{s-1}(\alpha_j-2D\tau)}{\log k((s+1)r)}-3\varepsilon\right).$$

Since τ is arbitrary and provided that ε is sufficiently small, we obtain

$$\begin{aligned}\liminf_{m\to+\infty}\frac{\log\|x_m\|}{\log k(m)}&\geq\liminf_{s\to+\infty}\left(\frac{\log\prod_{j=s_0}^{s-1}\alpha_j}{\log k((s+1)r)}-3\varepsilon\right)\\&=c-3\varepsilon>b.\end{aligned}$$

This establishes (3.13). Now we prove (3.14). We define

$$T=\limsup_{s\to+\infty}\frac{\|y_{sr}\|_{sr}}{\|z_{sr}\|_{sr}}.$$

By (3.24) we have $0\leq T\leq 1$. It follows from (3.21) and (3.24) that for all large s

$$\|z_{(s+1)r}\|_{(s+1)r}\geq(\alpha_s-2D\tau_s^1)\|z_{sr}\|_{sr}.$$

Together with (3.22), this yields that for all large s,

$$\frac{\|y_{(s+1)r}\|_{(s+1)r}}{\|z_{(s+1)r}\|_{(s+1)r}}\leq\frac{\beta_s+D\tau_s^2}{\alpha_s-2D\tau_s^1}\frac{\|y_{sr}\|_{sr}}{\|z_{sr}\|_{sr}}+\frac{D\tau_s^2}{\alpha_s-2D\tau_s^1}.$$

Taking the limit superior on both sides and using (3.18) and (3.20), we obtain $T\leq(\beta_s/\alpha_s)T$. Under the assumption (H3) we have

$$\lim_{s\to\infty}\frac{\beta_s}{\alpha_s}\leq\eta<1,$$

this implies $T=0$, and (3.14) holds. □

Proof of Theorem 3.1. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence satisfying the hypotheses of Theorem 3.1. If $x_{k'}=0$ for some k', then it follows from (3.3) that $x_k=0$ for all $k\geq k'$, and hence, the first alternative in the theorem holds. Now we assume that $x_k\neq 0$ for all $k\geq k'$. Let $\lambda_1<\cdots<\lambda_p$ be the Lyapunov exponents of the sequence $(A_m)_{m\in\mathbb{N}}$.

On both sides of λ_i, take real numbers b_j such that

$$\lambda_{j-1}<b_{j-1}<\lambda_j$$

and

$$\lambda_j<b_j<\lambda_{j+1}.$$

Take $b_0<\lambda_1$ when $\lambda_1\neq-\infty$ and $b_p>\lambda_p$.

Applying Lemma 3.4 to each number $b=b_j$, we conclude that there exists $j\in\{1,\ldots,p\}$ such that

$$\limsup_{m\to+\infty}\frac{\log\|x_m\|}{\log h(m)}<b_j$$

and

$$\liminf_{m\to+\infty}\frac{\log\|x_m\|}{\log k(m)}>b_{j-1}.$$

Considering $h(m)=k(m)$ and letting $b_j\searrow\lambda_j$ and $b_{j-1}\nearrow\lambda_j$, we find that

$$\lim_{m\to+\infty}\frac{\log\|x_m\|}{\log h(m)}=\lambda_j.$$

Now the proof is finished. □

4 Examples

In this section, we present the following examples which will show the (h,k,μ,ν)-dichotomies. To show the difference with different values of h, k, μ and ν, we follow the ideas of Naulin and Pinto in [20]. In order to make precise statements, we first introduce some notations and concepts for difference equations.

Now, we introduce the sequence spaces

$$l_h := \left\{ x\colon \mathbb{N} \to \mathbb{C}^m \mid \sup_{m\in\mathbb{N}} |h_m^{-1} x_m| < \infty \right\},$$
$$l_{h,0} := \left\{ x \in l_h \mid \lim_{m\to\infty} h_m^{-1} x_m = 0 \right\},$$

which equipped with the norm

$$\|x\|_h = \sup_{m\in\mathbb{N}} |h_m^{-1} x_m|.$$

It is easy to see that the spaces $(l_h, \|\cdot\|_h)$ and $(l_{h,0}, \|\cdot\|_h)$ are Banach spaces. Let V_h be the subspace of $\mathbb{C}^m$ defining by the following property: if $\xi \in V_h$, then the solution of the linear difference system (1.1) with initial condition $x_0 = \xi$ belongs to l_h. Analogously we introduce the subspace $V_{h,0}$ of the initial conditions by the following property: if $\xi \in V_{h,0}$, then the solution of the linear difference system (1.1) with initial condition $x_0 = \xi$ belongs to $l_{h,0}$.

Following the ideas of [20] (see also Chapter 2 in [15]), we consider the nonuniform behavior, and then we have the following property:

If (1.1) has the (h,k,μ,ν)-dichotomy

$$\|\mathcal{A}(m,l)P_l\| \le K\frac{h(m)}{h(l)}\mu(l), \qquad m \ge l, \tag{4.1}$$

and

$$\|\mathcal{A}(m,l)Q_l\| \le K\frac{k(m)}{k(l)}\nu(l), \qquad m \le l. \tag{4.2}$$

and (H1) is fulfilled, then

$$V_{h,0} \subset V_{k,0} \subset P[\mathbb{C}^m] \subset V_h \subset V_k,$$

where $P\colon \mathbb{C}^m \to \mathbb{C}^m$ is a projection such that $PP = P$.

Now two linear difference systems, which admit (h,k,μ,ν)-dichotomies but does not admit (h,k)-dichotomies, will be given to illustrate the relation of $V_{h,0}$, $V_{k,0}$, $P[\mathbb{C}^m]$, V_h and V_k.

Example 4.1. Now we consider the system

$$x_{m+1} = \begin{pmatrix} e^{-1+\frac{1}{4}m(-1)^m-\frac{1}{4}(m-1)(-1)^{m-1}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{1-\frac{1}{4}m(-1)^m+\frac{1}{4}(m-1)(-1)^{m-1}} \end{pmatrix} x_m \tag{4.3}$$

and define the projection matrices

$$P_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\mathcal{A}(m,l)P_1 = \begin{pmatrix} e^{-(m-l-1)+\frac{1}{4}(m-l-1)(-1)^{m-1}+\frac{1}{4}l(-1)^{(m-1)}+\frac{1}{4}l(-1)^{l}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{A}(m,l)P_2 = \begin{pmatrix} e^{-(m-l-1)+\frac{1}{4}(m-l-1)(-1)^{m-1}+\frac{1}{4}l(-1)^{(m-1)}+\frac{1}{4}l(-1)^{l}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then (4.1) holds with

$$K = e^{\frac{5}{4}}, \qquad h(m) \geq e^{-\frac{3}{4}m}, \quad \text{and} \quad \mu(m) \geq e^{\frac{1}{2}m},$$

and (4.2) holds with

$$K = e^{\frac{5}{4}}, \qquad k(m) \leq e^{\frac{3}{4}m}, \quad \text{and} \quad \nu(m) \geq e^{\frac{1}{2}m}.$$

Besides, it is easy to verify that the nonuniform part can not be removed, see [24] for details. Thus we can list the following (h,k,μ,ν)-dichotomies:

D1′. With projection $P = P_1$ the system (4.3) has an (h,k,μ,ν)-dichotomy with $h(m) = k(m) = 1$, $\mu(m) = \nu(m) = e^{\frac{1}{2}m}$ and the property $V_{h,0} = V_{k,0} \neq P[\mathbb{C}^m] = V_h = V_k$.

D2′. With projection $P = P_2$ the system (4.3) has an (h,k,μ,ν)-dichotomy with $h(m) = k(m) = 1$, $\mu(m) = \nu(m) = e^{\frac{1}{2}m}$ and the property $V_{h,0} = V_{k,0} = P[\mathbb{C}^m] \neq V_h = V_k$.

D3′. With projection $P = P_1$ the system (4.3) has an (h,k,μ,ν)-dichotomy with $h(m) = 1$, $k(m) = e^{\frac{3}{4}m}$, $\mu(m) = \nu(m) = e^{\frac{1}{2}m}$ and the property $V_{h,0} \neq V_{k,0} = P[\mathbb{C}^m] = V_h \neq V_k$.

D4′. With projection $P = P_1$ the system (4.3) has an (h,k,μ,ν)-dichotomy with $h(m) = k(m) = e^{\frac{3}{4}m}$, $\mu(m) = \nu(m) = e^{\frac{1}{2}m}$ and the property $V_{h,0} = V_{k,0} = P[\mathbb{C}^m] \neq V_h = V_k$.

Example 4.2. Now we consider the system

$$x_{m+1} = \begin{pmatrix} \frac{m+1}{m}e^{-1+\frac{1}{4}m(-1)^m-\frac{1}{4}(m-1)(-1)^{m-1}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{m+1}{m}e^{1-\frac{1}{4}m(-1)^m+\frac{1}{4}(m-1)(-1)^{m-1}} \end{pmatrix} x_m \tag{4.4}$$

with projections P_1, P_2 defined in the Example 4.1.

Since

$$\mathcal{A}(m,l)P_1 = \begin{pmatrix} \frac{m}{l}e^{-(m-l-1)+\frac{1}{4}(m-l-1)(-1)^{m-1}+\frac{1}{4}l(-1)^{(m-1)}+\frac{1}{4}l(-1)^{l}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{A}(m,l)P_2 = \begin{pmatrix} \frac{m}{l}e^{-(m-l-1)+\frac{1}{4}(m-l-1)(-1)^{m-1}+\frac{1}{4}l(-1)^{(m-1)}+\frac{1}{4}l(-1)^{l}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then (4.1) holds with

$$K = e^{\frac{5}{4}}, \qquad h(m) \geq me^{-\frac{3}{4}m}, \quad \text{and} \quad \mu(m) \geq e^{\frac{1}{2}m},$$

and (4.2) holds with

$$K = e^{\frac{5}{4}}, \qquad k(m) \leq me^{\frac{3}{4}m}, \quad \text{and} \quad \nu(m) \geq e^{\frac{1}{2}m}.$$

Thus we can list the following (h,k,μ,ν)-dichotomies:

D1″. With projection $P = P_1$ the system (4.4) has an (h,k,μ,ν)-dichotomy with $h(m) = k(m) = m$, $\mu(m) = \nu(m) = e^{\frac{1}{2}m}$ and the property $V_{h,0} = V_{k,0} = P[\mathbb{C}^m] = V_h = V_k$.

D2″. With projection $P = P_1$ the system (4.4) has an (h,k,μ,ν)-dichotomy with $h(m) = k(m) = me^{\frac{3}{4}m}$, $\mu(m) = \nu(m) = e^{\frac{1}{2}m}$ and the property $V_{h,0} = V_{k,0} = P[\mathbb{C}^m] \neq V_h = V_k$.

Remark 4.3. From the analysis above, it is easy to verify that hypotheses (H1)–(H3) can be satisfied in D4′ of Example 4.1 and D2″ of Example 4.2 respectively, and, consequently, Theorem 3.1 are applicable to these examples.

Acknowledgements

The authors would like to thank the anonymous referees and Professor Jifeng Chu for their careful reading and valuable comments for revising the paper. Hailong Zhu was supported by the National Natural Science Foundation of China (NO. 11301001), Excellent Youth Scholars Foundation and the Natural Science Foundation of Anhui Province of PR China (NO. 2013SQRL030ZD, NO. 1208085QG131, NO. KJ2014A003, NO. 1508085QA13). Yongxin Jiang was partly supported by the National Natural Science Foundation of China (NO. 11401165).

References

[1] L. Barreira, J. Chu, C. Valls, Robustness of nonuniform dichotomies with different growth rates, *São Paulo J. Math. Sci.* **5**(2011), 203–231. MR2987797

[2] L. Barreira, J. Chu, C. Valls, Lyapunov functions for general nonuniform dichotomies, *Milan J. Math.* **81**(2013), 153–169. MR3046985

[3] L. Barreira, Ya. Pesin, *Lyapunov exponents and smooth ergodic theory*, University Lecture Series, Vol. 23, American Mathematical Society, Providence, RI, 2002. MR1862379

[4] L. Barreira, C. Valls, Stability of nonautonomous differential equations in Hilbert spaces, *J. Differential Equations* **217**(2005), 204–248. MR2170533

[5] L. Barreira, C. Valls, *Stability of nonautonomous differential equations*, Lecture Notes in Mathematics, Vol. 1926, Springer, 2008. MR2368551

[6] L. Barreira, C. Valls, Robustness of general dichotomies, *J. Funct. Anal.* **257**(2009), 464–484. MR2527025

[7] L. Barreira, C. Valls, Nonautonomous difference equations and a Perron-type theorem, *Bull. Sci. Math.* **136**(2012), 277–290. MR2914948

[8] L. Barreira, C. Valls, Nonautonomous equations with arbitrary growth rates: a Perron-type theorem, *Nonlinear Anal.* **75**(2012), 6203–6215. MR2956137

[9] L. Barreira, C. Valls, A Perron-type theorem for nonautonomous difference equations, *Nonlinearity* **26**(2013), 855–870. MR3033573

[10] L. Barreira, C. Valls, On the exponential behaviour of non-autonomous difference equations, *Proc. Edinb. Math. Soc.* **56**(2013), 643–656. MR3109751

[11] A. J. G. Bento, C. Silva, Nonuniform (μ, ν)-dichotomies and local dynamics of difference equations, *Nonlinear Anal.* **75**(2012), 78–90. MR2846783

[12] J. Chu, Robustness of nonuniform behavior for discrete dynamics, *Bull. Sci. Math.* **137**(2013), 1031–1047. MR3130345

[13] C. Coffman, Asymptotic behavior of solutions of ordinary difference equations, *Trans. Amer. Math. Soc.* **110**(1964), 22–51. MR0156122

[14] W. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath and Co, 1965. MR0190463

[15] W. Coppel, *Dichotomy in stability theory*, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, New York/Berlin, 1978. MR0481196

[16] S. Elaydi, *An introduction to difference equations*, Springer, New York, 2005. MR2128146

[17] P. Hartman, A. Wintner, Asymptotic integrations of linear differential equations, *Amer. J. Math.* **77**(1955), 45–86. MR0066520

[18] J. Massera, J. Schäffer, *Linear differential equations and function spaces*, Pure and Applied Mathematics, Vol. 21, Academic Press, 1966. MR0212324

[19] K. Matsui, H. Matsunaga, S. Murakami, Perron type theorem for functional differential equations with infinite delay in a Banach space, *Nonlinear Anal.* **69**(2008), 3821–3837. MR2463337

[20] R. Naulin, M. Pinto, Roughness of (h, k) dichotomies, *J. Differential Equations* **118**(1995), 20–35. MR1329401

[21] O. Perron, Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen, *Math. Z.* **29**(1929), 129–160. MR1544998

[22] M. Pituk, A Perron type theorem for functional differential equations, *J. Math. Anal. Appl.* **316**(2006), 24–41. MR2201747

[23] M. Pituk, Asymptotic behavior and oscillation of functional differential equations, *J. Math. Anal. Appl.* **322**(2006), 1140–1158. MR2250641

[24] J. Chu, H. Zhu, S. Siegmund, Y. Xia, Nonuniform dichotomy spectrum and reducibility for nonautonomous difference equations, preprint, arXiv:1402.1599 [math.DS].